Optimal Risk Probability for First Passage Models

—in Semi-Markov Decision Processes

Xianping Guo (Coauthor: Yonghui Huang)
(Zhongshan University, Guangzhou)

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Outline

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1. Motivation

Background: Reliability engineering, and risk analysis

Problem: $\sup_\pi P_i^\pi(\tau_B > \lambda)$,

- i an initial state
- π is a policy
- B is a given target set
- τ_B is a first passage time to B
- λ is a threshold value.

2. Semi-Markov Decision Processes

The model of SMDP:

$$\{S, B, (A(i), i \in S), Q(t, j|i, a)\}$$

where

- S : the state space, a denumerable set;
- B: a given target set, a subset of S;
- $A(i)$: finite set of actions available at $i \in S$;
- $Q(t, j|i, a)$: semi-Markov kernel, $a \in A(i), i, j \in S$;

Notation:

- Policy π: A sequence $\pi = \{\pi_n, n = 0, 1, \ldots\}$ of stochastic kernels π_n on the action space A given H_n satisfying

  $\pi_n(A(i_n)|(0, i_0, \lambda_0, a_0, \ldots, t_{n-1}, i_{n-1}, \lambda_{n-1}, a_{n-1}, t_n, i_n) = 1$

- Stationary policy: measurable f, $f(i, \lambda) \in A(i)$ for all (i, λ)

- $P^{\pi}_{(i,\lambda)}$: Probability measure on $(S \times [0, \infty) \times (\cup_{i \in S} A(i)))^{\infty}$

- S_n, J_n, A_n: n-th decision epoch, the state and action at the S_n, respectively.

Assumption A. There exist $\delta > 0$ and $\epsilon > 0$ such that

$$\sum_{j \in S} Q(\delta, j|i, a) \leq 1 - \epsilon, \text{ for all } (i, a) \in K.$$

Assumption A $\Rightarrow P^{\pi}_{(i,\lambda)}(\{S_\infty = \infty\}) = 1$

Semi-Markov decision process $\{(Z(t), A(t), t \geq 0\}$:

$$Z(t) = J_n, A(t) = A_n, \quad \text{for} \quad S_n \leq t < S_{n+1}$$

The first passage time into B, is defied by

$$\tau_B := \inf\{t \geq 0 \mid Z(t) \in B\}, \quad (\text{with} \ \inf \emptyset := \infty),$$

3. Optimality Problems

The risk probability:

$$F^{\pi}(i, \lambda) := P^{\pi}_{(i,\lambda)}(\tau_B \leq \lambda)$$

The optimal value:

$$F_*(i, \lambda) := \inf_{\pi \in \Pi} F^{\pi}(i, \lambda),$$

Definition 1. A policy $\pi^* \in \Pi$ is called optimal if

$$F^{\pi^*}(i, \lambda) = F_*(i, \lambda) \;\; \forall \, (i, \lambda) \in S \times R.$$

- Existence and computation of optimal policies ???

4. Optimality Equation

For $i \in B^c, a \in A(i)$, and $\lambda \geq 0$, let

$$T^a u(i,\lambda) := Q(\lambda, B|i,a) + \sum_{j \in B^c} \int_0^\lambda Q(dt, j|i,a) u(j, \lambda - t),$$

with $u \in \mathcal{F}_{[0,1]}$ (the set of measurable functions $0 \leq u \leq 1$),

$$Q(\lambda, B|i,a) := \sum_{j \in B} Q(\lambda, j|i,a), \quad T^a u(i,\lambda) := 0 \text{ for } \lambda < 0.$$

Then, define operators T and T^f:

$$Tu(i,\lambda) := \min_{a \in A(i)} T^a u(i,\lambda); \quad T^f u(i,\lambda) := T^{f(i,\lambda)} u(i,\lambda),$$

for each stationary policy f.

Theorem 1. Let Under Assumption A, we have

(a) $F^f = \lim_{n\to\infty} u_n^f$, where $u_n^f := T^f u_{n-1}, u_{-1}^f := 1$;

(b) F^f satisfied the equation, $u = T^f u$, for all $f \in F$;

- Theorem 1 gives an approximation of risk probability F^f.

For each $(i, \lambda) \in B^c \times R_+$ and $\pi \in \Pi$, let

$$F_{-1}^{\pi}(i, \lambda) := 1,$$

$$F_n^{\pi}(i, \lambda) := 1 - \sum_{m=0}^{n} P_{(i,\lambda)}^{\pi}(S_m \leq \lambda < S_{m+1}, J_k \in B^c, 0 \leq k \leq m)$$

Theorem 2. Let $F_n^*(i,\lambda) := \inf_{\pi} F_n^{\pi}(i,\lambda)$, then

(a) $F_{n+1}^* = TF_n^*$ for all $n \geq -1$, and $\lim_{n\to\infty} F_n^* = F_*$.

(b) F_* satisfies the optimality equation: $F_* = TF_*$.

(c) F_* is the maximal fixed point of T in $\mathcal{F}_{[0,1]}$.

Remark 1.

- Theorem 2(a) gives a value iteration algorithm for computing the optimal value function F_*.
- Theorem 2(b) establishes the optimality equation.

5. Existence of Optimality Policise

To ensure the existence of optimal policies, we introduce the following condition.

Assumption B. For every $(i, \lambda) \in B^c \times R$ and f,

$$P^f_{(i,\lambda)}(\tau_B < \infty) = 1.$$

To verify Assumption B, we have a fact below:

Theorem 3. If there exists a constant $\alpha > 0$ such that

$$\sum_{j \in B} Q(\infty, j|i, a) \geq \alpha \text{ for all } i \in B^c, a \in A(i),$$

then Assumption B holds.

Theorem 4. Under Assumptions A and B, we have

(a) F^f and F_* are the unique solution in $\mathcal{F}_{[0,1]}$ to equations $u = T^f u$ and $u = Tu$, respectively;

(b) any f, such that $F_* = T^f F_*$, is optimal;

(c) there exists a stationary policy f^* satisfying the optimality equation: $F_* = TF_* = T^{f^*} F_*$, and such policy f^* is optimal.

Remark 2.

- Theorem 4(c) shows the existence of an optimal poliy.

To give the existence of special optimal policies, let

$$A^*(i,\lambda) := \{a \in A(i) \mid F^*(i,\lambda) = T^a F^*(i,\lambda)\}.$$
$$A^*(i) := \bigcap_{\lambda \geq 0} A^*(i,\lambda)$$

Theorem 5. If $\sup_{i \in B^c} \sup_{a \in A(i)} Q(t, B^c \mid i, a) < 1$ for some $t > 0$, and Assumptions A and B hold, then,

(a) for any $g \in G := \{g| \ g(i) \in A(i) \forall i \in S\}$, F^g is the unique solution in $\mathcal{F}_{[0,1]}$ to the equation: $u = T^g u$;

(b) there exists an optimal policy $f \in G$ if and only if $A^*(i) \neq \emptyset$ for all $i \in B^c$.

5. Numerable examples

Example 5.1. Let $S = \{1, 2, 3\}$, B=$\{3\}$, where

- state 1: the good state
- state 2: the medium state
- state 3: the failure state

Let $A(1) = \{a_{11}, a_{12}\}, A(2) = \{a_{21}, a_{22}\}, A(3) = \{a_{31}\}$.

The semi-Markov kernel is of the form:

$$Q(t, j \mid i, a) = H(t \mid i, a)p(j \mid i, a)$$

- $H(t \mid i, a)$: the distribution functions of the sojourn time
- $p(j \mid i, a)$: the transition probabilities.

$$H(t \mid 1, a_{11}) := \begin{cases} 1/25, & t \in [0, 25], \\ 1, & t > 25; \end{cases}$$

$$H(t \mid 2, a_{21}) := \begin{cases} 1/20, & t \in [0, 20], \\ 1, & t > 20; \end{cases}$$

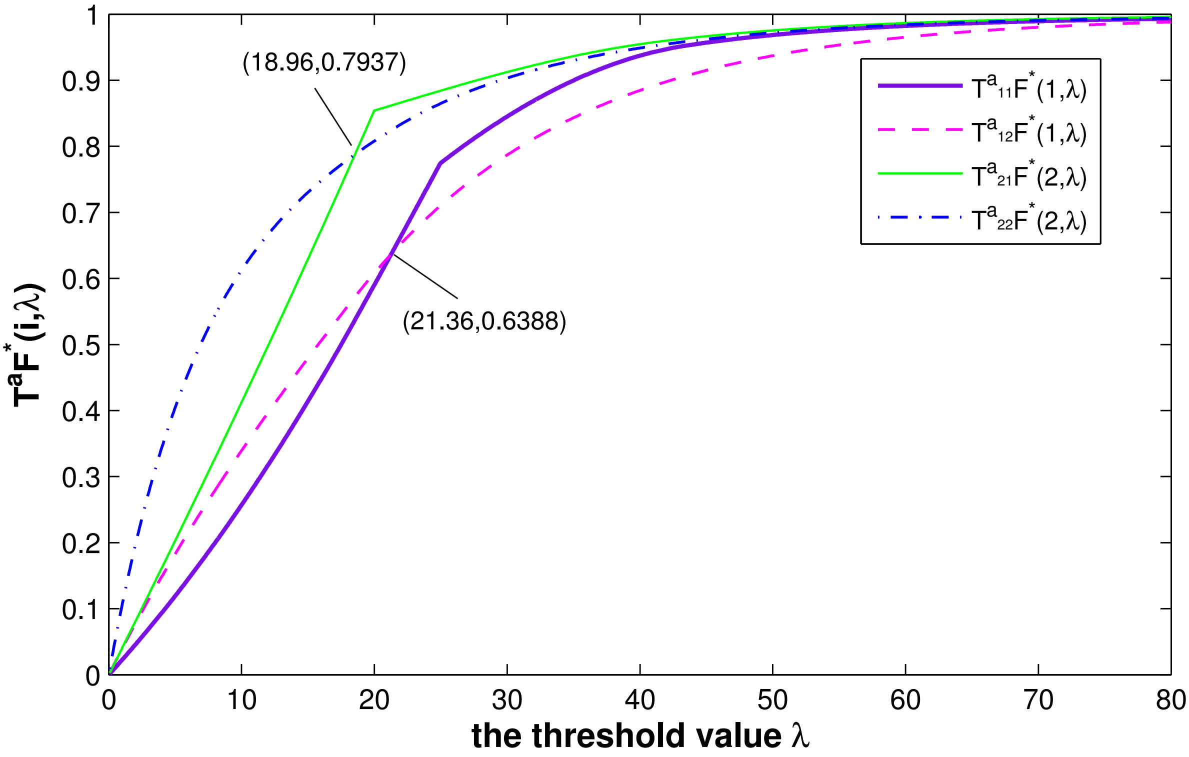
$$H(t \mid 3, a_{31}) := 1 - e^{-0.2t}.$$

$$H(t \mid 1, a_{12}) = 1 - e^{-0.08t};$$

$$H(t \mid 2, a_{22}) = 1 - e^{-0.15t};$$

$$p(1 \mid 1, a_{11}) = 0,\ p(2 \mid 1, a_{11}) = \frac{9}{20},\ p(3 \mid 1, a_{11}) = \frac{11}{20};$$
$$p(1 \mid 1, a_{12}) = 0,\ p(2 \mid 1, a_{12}) = \frac{1}{2},\ p(3 \mid 1, a_{12}) = \frac{1}{2};$$
$$p(1 \mid 2, a_{21}) = \frac{1}{5},\ p(2 \mid 2, a_{21}) = 0,\ p(3 \mid 2, a_{21}) = \frac{4}{5};$$
$$p(1 \mid 2, a_{22}) = \frac{1}{4},\ p(2 \mid 2, a_{22}) = 0,\ p(3 \mid 2, a_{22}) = \frac{3}{4};$$
$$p(3 \mid 3, a_{31}) = 1.$$

Using the value iteration algorithm in Theorem 2, we obtain some computational results as in Figure 1 and Figure 2.



(18.96,0.7937)
(21.36,0.6388)
$T^a{}_{11}F^*(1,\lambda)$
$T^a{}_{12}F^*(1,\lambda)$
$T^a{}_{21}F^*(2,\lambda)$
$T^a{}_{22}F^*(2,\lambda)$
$T^aF^*(i,\lambda)$
the threshold value λ
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
10
20
30
40
50
60
70
80

Figure 1. The functions $T^a F^*(i, \lambda)$

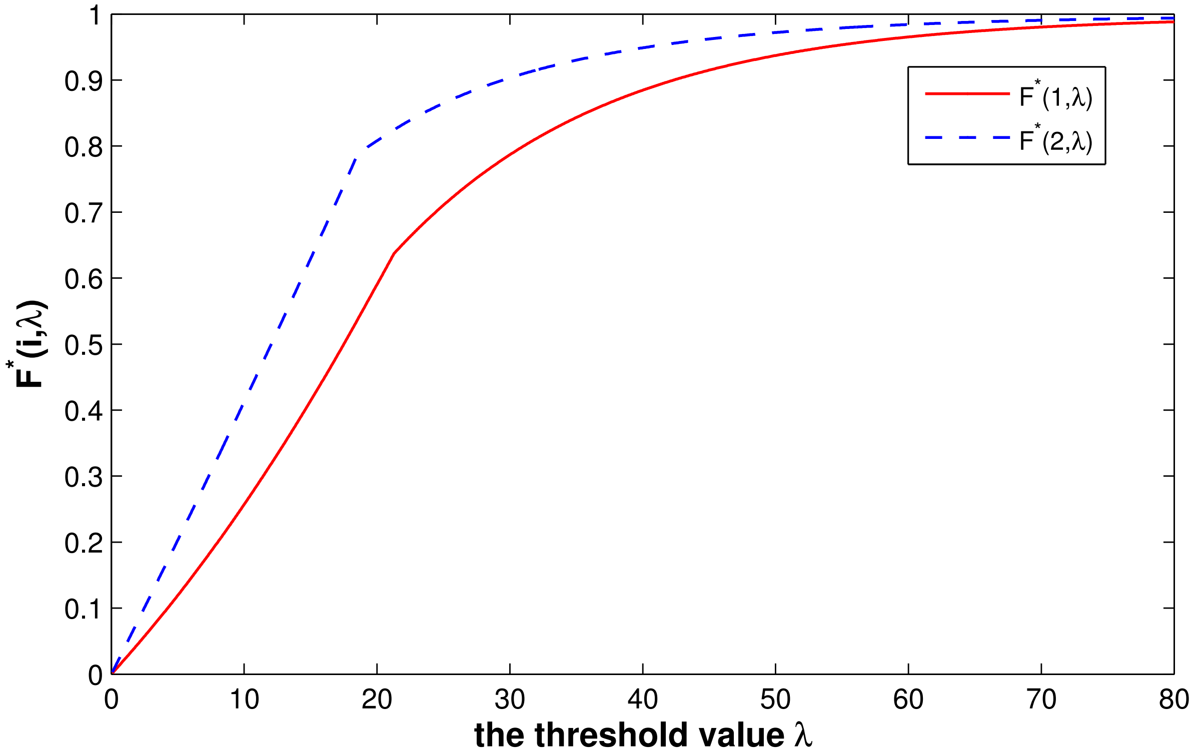


Figure 2. The value function $F^*(i, \lambda)$

More clearly, we have

$$F^*(1,\lambda) \;=\; \begin{cases} T^{a_{11}}F^*(1,\lambda), & 0 \le \lambda < 21.36, \\ T^{a_{11}}F^*(1,\lambda) = T^{a_{12}}F^*(1,\lambda), & \lambda = 21.36, \\ T^{a_{12}}F^*(1,\lambda), & 21.36 < \lambda < 29.3, \\ T^{a_{11}}F^*(1,\lambda) = T^{a_{12}}F^*(1,\lambda), & \lambda = 29.3, \\ T^{a_{11}}F^*(1,\lambda)(= 0.7742), & \lambda > 29.3, \end{cases}$$

$$F^*(2,\lambda) \;=\; \begin{cases} T^{a_{21}}F^*(2,\lambda), & 0 \le \lambda < 18.54, \\ T^{a_{21}}F^*(2,\lambda) = T^{a_{22}}F^*(2,\lambda), & \lambda = 18.54, \\ T^{a_{22}}F^*(2,\lambda), & 18.54 < \lambda < 23.82, \\ T^{a_{21}}F^*(2,\lambda) = T^{a_{22}}F^*(2,\lambda), & \lambda = 23.82, \\ T^{a_{21}}F^*(2,\lambda)(= 0.8542), & \lambda > 23.82. \end{cases}$$

Define a policy f^* by

$$f^*(1,\lambda) = \begin{cases} a_{11}, & 0 \le \lambda \le 21.36, \\ a_{12}, & 21.36 < \lambda \le 29.3, \\ a_{11}, & \lambda > 29.3, \end{cases}$$

$$f^*(2,\lambda) = \begin{cases} a_{21}, & 0 \le \lambda \le 18.54, \\ a_{22}, & 18.54 < \lambda \le 23.82, \\ a_{21}, & \lambda > 23.82, \end{cases}$$

Then, we have

- $F^*(i,\lambda) = T^{f^*}F^*(i,\lambda)$ for $i = 1, 2$ and all $\lambda \ge 0$,
- f^* is an optimal stationary policy.

$$A^*(1,\lambda) = \begin{cases} \{a_{11}\}, & 0 \le \lambda < 21.36, \\ \{a_{11}, a_{12}\}, & \lambda = 21.36, \\ \{a_{12}\}, & 21.36 < \lambda < 29.3, \\ \{a_{11}, a_{12}\}, & \lambda = 29.3, \\ \{a_{11}\}, & \lambda > 29.3, \end{cases}$$

$$A^*(2,\lambda) = \begin{cases} \{a_{21}\}, & 0 \le \lambda < 18.54, \\ \{a_{21}, a_{22}\}, & \lambda = 18.54, \\ \{a_{22}\}, & 18.54 < \lambda < 23.82, \\ \{a_{21}, a_{22}\}, & \lambda = 23.82, \\ \{a_{21}\}, & \lambda > 23.82, \end{cases}$$

Hence,

$$A^*(1) = \bigcap_{\lambda \geq 0} A^*(1, \lambda) = \emptyset, A^*(2) = \bigcap_{\lambda \geq 0} A^*(2, \lambda) = \emptyset,$$

which show there is no optimal policy in G.

Remark 3. This shows that the assumption in the previous literature is not satisfied for this example !!!

Example 5.2. Let $S = \{1, 2\}$, B=$\{2\}$;

$A(1) = \{a_{11}, a_{12}\}, A(2) = \{a_{21}\}$;

$Q(t, j \mid i, a)$ is given by

$$Q(t, j \mid 1, a_{11}) = \begin{cases} 1/2, & \text{if } t \geq 1, j = 1, 2, \\ 0, & \text{otherwise;} \end{cases}$$

$$Q(t, j \mid 1, a_{12}) = \begin{cases} 1, & \text{if } t \geq 2, j = 2, \\ 0, & \text{otherwise;} \end{cases}$$

$$Q(t, j \mid 2, a_{21}) = \begin{cases} 1 - e^{-t}, & \text{if } t \geq 0, j = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Assumptions A and B holds in this example.

We now define a policy d as follows:

$$d(1,\lambda) = \begin{cases} a_{12}, & 0 \le \lambda \le 2, \\ a_{11}, & \lambda > 2. \end{cases}$$

Then, by Theorem 1, we have $F^d(1,\lambda) = \lim_{n\to\infty} F_n^d(1,\lambda)$, which yields

$$F^d(1,\lambda) = \begin{cases} 0, & 0 \le \lambda < 2, \\ 1, & \lambda = 2, \\ 1/2, & 2 < \lambda < 3. \end{cases}$$

Hence, $F^d(1,\lambda)$ is not a distribution function of λ.

Many Thanks !!!